

Isolated singularities of affine special Kähler metrics in two dimensions

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Abstract

We prove that there are just two types of isolated singularities of special Kähler metrics in real dimension two provided the associated holomorphic cubic form does not have essential singularities. We also construct examples of such metrics.

1 Introduction

Special Kähler metrics attracted a lot of interest recently both in mathematics and physics, see for instance [ACDM15, MS15, Nei14] for the most recent works. The main source of interest to such metrics is the fact that the total space of the cotangent bundle of the underlying manifold carries a natural hyperKähler metric for which each fiber is a Lagrangian submanifold. Such manifolds play an important role in the SYZ-conjecture [SYZ96].

Soon after special Kähler metrics entered the mathematical scene [Fre99], Lu proved [Lu99] that there are no complete special Kähler metrics besides flat ones. This motivates studying singular special Kähler metrics as the natural structure on bases of holomorphic Lagrangian fibrations with singular fibers. In this paper we study isolated singularities of affine special Kähler metrics in the lowest possible dimension.

Recall that a Kähler manifold (M, g, I, ω) is called (affine) special Kähler, if it is equipped with a symplectic, torsion-free, flat connection ∇ such that

$$(\nabla_X I)Y = (\nabla_Y I)X \quad (1)$$

for all tangent vectors X and Y . To any special Kähler metric one can associate a holomorphic cubic form Ξ , which measures the difference between the Levi-Civita connection and ∇ .

Throughout the rest of this paper we assume that $\dim_{\mathbb{R}} M = 2$, i.e. M is a Riemann surface. Let m_0 be an isolated singularity of g . Denote by n the order of Ξ at m_0 , i.e. m_0 is a zero of order n if $n > 0$ or m_0 is a pole of order $|n|$ if $n < 0$ or $\Xi(m_0)$ exists

and does not vanish if $n = 0$. By choosing a holomorphic coordinate z near m_0 , we can assume that g is a special Kähler metric on the punctured disc $B_1^* = B_1(0) \setminus \{0\}$. The following is the main result of this paper.

Theorem 1.1. *Let $g = w|dz|^2$ be a special Kähler metric on B_1^* . Assume that Ξ is holomorphic on the punctured disc and the order of Ξ at the origin is $n > -\infty$. Then*

$$w = -|z|^{n+1} \log |z| e^{O(1)} \quad \text{or} \quad w = |z|^\beta (C + o(1)) \quad (2)$$

as $z \rightarrow 0$, where $C > 0$ and $\beta < n + 1$.

Moreover, for any $n \in \mathbb{Z}$ and $\beta \in \mathbb{R}$ such that $\beta < n + 1$ there is an affine special Kähler metric satisfying (2).

We prove this theorem by establishing an intimate relation between special Kähler metrics and metrics of non-positive Gaussian curvature and applying the machinery developed for the latter ones. This relation allows us in particular to construct explicit examples of special Kähler metrics from metrics of *constant* negative curvature.

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2 Affine special Kähler metrics in local coordinates

Let $\Omega \subset \mathbb{C}$ be an open subset, which we equip with the flat metric $g_0 = |dz|^2 = dx^2 + dy^2$. Denote by $*$ the corresponding Hodge operator and by $\Delta = \partial_{xx}^2 + \partial_{yy}^2$ the Laplace operator.

Proposition 2.1. *For any pair $(u, \eta) \in C^\infty(\Omega) \times \Omega^1(\Omega)$ satisfying*

$$d\eta = 0, \quad (3)$$

$$*d*\eta = 2*(*\eta \wedge du) - 2e^u|\eta|^2, \quad (4)$$

$$\Delta u = |2\eta + e^{-u}du|^2 e^{2u}. \quad (5)$$

the metric $g = e^{-u}(dx^2 + dy^2)$ is special Kähler. Conversely, for any special Kähler metric on Ω there exists a solution (u, η) of (3)-(5).

Proof. In real coordinates (x, y) the connection ∇ can be represented by

$$\omega_\nabla = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \in \Omega^1(\mathbb{R}^2; \mathfrak{gl}_2(\mathbb{R})).$$

Then (1) can be written as $[\omega(X), I_0](Y) = [\omega(Y), I_0]X$, where

$$I_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since this equation is symmetric with respect to X and Y , it is enough to check its validity for $(X, Y) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, which yields

$$\begin{aligned}\omega_{11}(\frac{\partial}{\partial x}) - \omega_{22}(\frac{\partial}{\partial x}) &= -(\omega_{12}(\frac{\partial}{\partial y}) + \omega_{21}(\frac{\partial}{\partial y})), \\ \omega_{12}(\frac{\partial}{\partial x}) + \omega_{21}(\frac{\partial}{\partial x}) &= \omega_{11}(\frac{\partial}{\partial y}) - \omega_{22}(\frac{\partial}{\partial y}).\end{aligned}\tag{6}$$

Write $\omega_{11} - \omega_{22} = a dx + b dy$. It follows from (6) that $\omega_{12} + \omega_{21} = b dx - a dy$. Denoting $\omega_{11} + \omega_{22} = p dx + q dy$, we obtain

$$\omega_{11} = \frac{p+a}{2}dx + \frac{q+b}{2}dy \quad \text{and} \quad \omega_{22} = \frac{p-a}{2}dx + \frac{q-b}{2}dy.\tag{7}$$

Furthermore, the torsion of ∇ vanishes if and only if $\omega_{12}(\frac{\partial}{\partial x}) = \omega_{11}(\frac{\partial}{\partial y})$ and $\omega_{21}(\frac{\partial}{\partial y}) = \omega_{22}(\frac{\partial}{\partial x})$. Combing this with (7) we obtain

$$\omega_{12} = \frac{q+b}{2}dx - \frac{p+a}{2}dy \quad \text{and} \quad \omega_{21} = \frac{b-q}{2}dx + \frac{p-a}{2}dy,$$

which yields

$$\omega_{\nabla} = \begin{pmatrix} \omega_{11} & -*\omega_{11} \\ *\omega_{22} & \omega_{22} \end{pmatrix}.$$

Then ∇ is flat if and only if

$$\begin{aligned}d\omega_{11} &= \omega_{11} \wedge \omega_{22}, & d*\omega_{11} &= -*\omega_{11} \wedge \omega_{22} - |\omega_{11}|^2 dx \wedge dy, \\ d\omega_{22} &= \omega_{22} \wedge \omega_{11}, & d*\omega_{22} &= -*\omega_{22} \wedge \omega_{11} - |\omega_{22}|^2 dx \wedge dy.\end{aligned}\tag{8}$$

Here we used a special property of 1-forms in dimension 2, namely the identity $(*\alpha) \wedge (*\beta) = \alpha \wedge \beta$.

Furthermore, notice that ∇ preserves the Kähler form $\omega = 2e^{-u}dx \wedge dy$ if and only if

$$de^{-u} = d(\omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})) = e^{-u}(\omega_{11} + \omega_{22}) \quad \Leftrightarrow \quad -du = \omega_{11} + \omega_{22}.$$

Substituting this in (8) we obtain

$$\begin{aligned}d\omega_{11} &= du \wedge \omega_{11}, \\ d*\omega_{11} &= *\omega_{11} \wedge du - 2|\omega_{11}|^2 dx \wedge dy, \\ \Delta u &= -4*(*\omega_{11} \wedge du) + 4|\omega_{11}|^2 + |du|^2 = |2\omega_{11} + du|^2.\end{aligned}$$

Finally, substitute $\eta = e^{-u}\omega_{11}$ to obtain (3)-(5). \square

Observe that (5) implies that the Gaussian curvature of $\tilde{g} = e^{2u}|dz|^2$ equals $-|2\eta + e^{-u}du|^2$ and therefore is non-positive. This relation between special Kähler metrics and metrics of non-positive Gaussian curvature is central for the rest of the paper and will be more vivid below.

Corollary 2.2. *Any pair of functions (h, u) satisfying*

$$\Delta h = 0 \quad \text{and} \quad \Delta u = |dh|^2 e^{2u}, \quad (9)$$

on a domain $\Omega \subset \mathbb{R}^2$ determines a special Kähler metric on Ω . Conversely, if $H^1(\Omega; \mathbb{R})$ is trivial, then any special Kähler metric on Ω determines a solution of (9).

Proof. Assume $\eta = df$, which is always the case provided $H^1(\Omega; \mathbb{R})$ is trivial. By (4) we obtain

$$\Delta f = 2 * (*df \wedge du) - 2e^u |df|^2.$$

Denoting $h = 2f - e^{-u}$, we compute:

$$\begin{aligned} \Delta h &= 2\Delta f - e^{-u}(-\Delta u + |du|^2) \\ &= 4 * (*df \wedge du) - 4e^u |df|^2 - e^{-u}(-4e^{2u} |df|^2 + 4e^u * (*df \wedge du)) \\ &= 0. \end{aligned}$$

Hence, (h, u) solves (9).

Conversely, for any solution (h, u) of (9) the pair (u, df) solves (3)-(5), where $f = (h + e^{-u})/2$. \square

Observe that if h is a constant function, then (9) reduces to $\Delta u = 0$, i.e., the corresponding special Kähler metric $g = e^{-u}(dx^2 + dy^2)$ is flat. Non-trivial examples will be constructed below.

Corollary 2.3. *Assume $\Omega = B_1^*$. Then any triple (h, u, a) satisfying*

$$\Delta h = 0 \quad \text{and} \quad \Delta u = |dh + a\varphi|^2 e^{2u}, \quad (10)$$

where $a \in \mathbb{R}$ and φ is a 1-form generating $H^1(B_1^; \mathbb{R})$, determines a special Kähler metric on the punctured disc. Conversely, any special Kähler metric on the punctured disk determines a solution of (10).*

Proof. Recall that

$$\varphi = \frac{ydx - xdy}{x^2 + y^2}$$

is harmonic 1-form generating $H^1(B_1^*; \mathbb{R})$. Hence, any closed $\eta \in \Omega(B_1^*)$ can be written in the form $\eta = df + \frac{a}{2}\varphi$ for some $a \in \mathbb{R}$. Just like in the proof of Corollary 2.2 put $h = 2f - e^{-u}$ to obtain (10). \square

Notice that the second equation of (10) (as well as (9)) is the celebrated Kazdan–Warner equation [KW74].

Remark 2.4. Tracing through the proof one easily sees that given a solution of (10) the corresponding special Kähler structure is given by

$$\begin{aligned} g &= e^{-u}(dx^2 + dy^2), \quad \omega_{\nabla} = \begin{pmatrix} \omega_{11} & - * \omega_{11} \\ * \omega_{22} & \omega_{22} \end{pmatrix}, \\ \omega_{11} &= \frac{e^u}{2}(dh + a\varphi) - \frac{1}{2}du, \quad \omega_{22} = -\frac{e^u}{2}(dh + a\varphi) - \frac{1}{2}du. \end{aligned} \quad (11)$$

Example 2.5. Let h be a positive harmonic function. It is straightforward to check that the pair $(h, -\log h)$ solves (9). Choosing $h = -\log |z|$ we obtain that $g = -\log |z| |dz|^2$ is a special Kähler metric on the punctured unit disc. Special Kähler metrics with logarithmic singularities were studied for instance in [GW00, Lof05].

Example 2.6. Pick any integer n and consider the harmonic function

$$h(x, y) = \begin{cases} \operatorname{Re} z^{n+1} & \text{if } n \neq -1, \\ \log |z| & \text{if } n = -1. \end{cases}$$

Clearly, there are some positive constants C_1 and C_2 such that $-C_1|z|^{2n} \leq -|dh|^2 \leq -C_2|z|^{2n}$. Choose a point $z_0 \in B_2(0) \setminus B_1(0)$ and a non-positive smooth function \tilde{K} satisfying

$$\begin{aligned} \tilde{K}|_{B_1(0)} &= -|dh|^2, & \tilde{K}|_{\mathbb{C} \setminus B_2(0)} &= -1, \\ -C'_1|z - z_0|^{1-2n} &\leq \tilde{K} \leq -C'_2|z - z_0|^{1-2n}, \end{aligned}$$

where C'_1 and C'_2 are some positive constants. Clearly, \tilde{K} can be extended as a smooth function to $\mathbb{CP}^1 \setminus \{z_0, 0\}$, where we think of \mathbb{CP}^1 as $\mathbb{C} \cup \{\infty\}$.

Let g_0 be a Riemannian metric on \mathbb{CP}^1 such that $g_0 = |dz|^2$ on $B_1(0)$. Denote by K_0 the Gaussian curvature of g_0 . By [McO93, Thm.II] for any $\beta \leq n+1$ there exists a solution u of $\Delta_{g_0} u + \tilde{K}e^{2u} = K_0$ such that

$$u = \begin{cases} -\beta \log |z| + c + o(1), & \text{if } \beta < n+1, \\ -(n+1) \log |z| - \log |\log |z|| + O(1) & \text{if } \beta = n+1, \end{cases} \quad \text{as } z \rightarrow 0.$$

(u has also a similar behavior near z_0 .) Thus, on $B_1(0)$ the pair (h, u) solves (9). In particular, $g = w|dz|^2 = e^{-u}|dz|^2$ is a special Kähler metric satisfying (2).

Proof of Theorem 1.1. Let $\pi^{(1,0)} \in \Omega^{1,0}(T_{\mathbb{C}}B_1^*)$ be the projection onto $T^{1,0}B_1^*$. Recall the definition of the holomorphic cubic form:

$$\Xi = -\omega(\pi^{(1,0)}, \nabla \pi^{(1,0)}).$$

Then, a direct computation yields

$$\Xi = \frac{e^{-u}}{4} (*(\omega_{11} - \omega_{22}) - i(\omega_{11} - \omega_{22})) dz^2.$$

Therefore, substituting (11) we obtain $\Xi = \Xi_0 dz^3$, where

$$\Xi_0 = \frac{1}{2} \left(\frac{a}{2z} - \frac{\partial h}{\partial z} i \right).$$

Notice that $16|\Xi_0|^2 = |dh + a\varphi|^2 =: -\tilde{K}$.

Denoting $w = e^{-u}$, we obtain $\Delta u = -\tilde{K}e^{2u}$ on the punctured disc. Under the hypothesis of this theorem there exist positive constants C_1 and C_2 such that the inequalities

$$-C_1|z|^{2n} \leq \tilde{K} \leq -C_2|z|^{2n} \quad (12)$$

hold on a (possibly smaller) punctured disc.

By [McO93, Appendix B] we obtain that

$$u = -\beta \log |z| + c + o(1) \quad \text{or} \quad u = -(n+1) \log |z| - \log |\log |z|| + O(1),$$

where $\beta < n+1$. Since $w = e^{-u}$, we obtain (2).

The existence of special Kähler metrics satisfying (2) has been established in Example 2.6. \square

We remark in passing that starting from a different perspective, Loftin [Lof05] utilized an equation equivalent to (5) to study special Kähler metrics on \mathbb{CP}^1 .

3 Metrics of constant negative curvature and further examples

In this section we construct more examples — in particular explicit — of special Kähler metrics from metrics of *constant negative Gaussian curvature*.

Proposition 3.1. *Let $\tilde{g} = e^{2u}|dz|^2$ be a metric of constant negative Gaussian curvature on a domain $\Omega \subset \mathbb{R}^2$. Then $g = e^{-u}|dz|^2$ is special Kähler.*

Proof. Since \tilde{g} has a constant negative scalar curvature, say -1 , we have $\Delta u = e^{2u}$. Put $h(x, y) = x$ and observe that the pair (h, u) satisfies (9). Hence, $g = e^{-u}|dz|^2$ is a special Kähler metric. \square

Example 3.2. Different incarnations of the Poicaré metric lead to differently looking special Kähler metrics, which are gathered in the following table:

Constant negative curvature metric	Domain	Special Kähler metric
$\tilde{g} = (\operatorname{Im} z)^{-2} dz ^2$	upper half-plane	$g = \operatorname{Im} z dz ^2$
$\tilde{g} = 4(1 - z ^2)^{-2} dz ^2$	unit disc	$g = \frac{1}{2}(1 - z ^2) dz ^2$
$\tilde{g} = (z \log z)^{-2} dz ^2$	punctured disc	$g = - z \log z dz ^2$

Table 1: Poincaré metrics and corresponding special Kähler metrics

The special Kähler metric appearing in the first row can be found in [Fre99, Rem. 1.20]. Particularly interesting for us is the one appearing in the last row, as this yields an example of special Kähler metric with an isolated singularity.

Example 3.3. Explicit local examples of metrics of constant negative Gaussian curvature with conical singularities can be found in [KR08, Ex.2.1]. For instance, if $\alpha \in (0, 1)$, then

$$\tilde{g} = \frac{1}{|z|^{2\alpha}} \left(\frac{1 - \alpha}{1 - |z|^{2(1-\alpha)}} \right)^2 |dz|^2$$

is such a metric. Hence,

$$g = \frac{1}{1 - \alpha} |z|^\alpha (1 - |z|^{2(1-\alpha)}) |dz|^2$$

is a special Kähler metric with conical singularity.

Example 3.4. It is a classical result of Picard [Pic93] that for any given $n \geq 3$ pairwise distinct points (z_1, \dots, z_n) in \mathbb{C} and any n real numbers $(\alpha_1, \dots, \alpha_n)$ such that $\alpha_j < 1$ and $\sum \alpha_j > 2$ there exists a metric of constant negative curvature \tilde{g} on $\mathbb{C} \setminus \{z_1, \dots, z_n\}$ satisfying $\tilde{g} = |z - z_j|^{-2\alpha_j} (c + o(1)) |dz|^2$ near z_j . Hence, the corresponding special Kähler metric g has a conical singularity near z_j :

$$g = |z - z_j|^{\alpha_j} (c + o(1)) |dz|^2.$$

Explicit examples of constant negative curvature — hence special Kähler — metrics on three times punctured complex plane can be found in [KRS11] and references therein.

Notice also that it is possible to allow $\alpha_i = 1$ changing the asymptotic behaviour correspondingly and to replace \mathbb{C} by \mathbb{P}^1 .

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